PMP

PRISON MATHEMATICS PROJECT

SIMIODE EXPO 2022

THE PRISON MATHEMATICS PROJECT NEWSLETTER

ITERATION 3 – THE SIMIODE SPECIAL EDITION

IN THIS ITERATION

Greetings fellow math lovers attending the SIMIODE Expo 2022! We hope you will enjoy this "iteration" of the Prison Mathematics Project newsletter! We celebrate inclusivity and diversity in the math community by reaching out to lovers of mathematics, whether incarcerated or not. Our writers, editors and contributors from around the globe have designed this newsletter to share with you our deep appreciation for the wonderful world of mathematics.

The aim of the Prison Mathematics Project is to invite participants to experience a culture that leads to desistance from crime through the transformative power of a life centered around a passion for mathematics.

ASK THE MATH GURU

Do numbers freak you out? Does math send you into meltdown-mode? Are you wondering why we need to learn this stuff ANYways?? Don't worry, The Math Guru is here to help you work through your math trauma, one problem at a time. Ask for advice, guidance, or just a good ol' pep talk! You got this!

VaneSSa Vakharia,
The Math Guru

Hello Vanessa Vakharia,

My name is Aaron, and I want to expand my knowledge in the area of mathematics, seeing that men lie, women lie, but numbers don't! I want to align myself with the thing I know to be an absolute truth: mathematics. My experience with math is limited, and I was hoping to grow to understand physics, which is a big leap from basic mathematical principles. If you can help me, I would greatly appreciate it and forever be in debt to you. Thank you for your time and considerations.

Sincerely,
Aaron

Aaron!

I LOVED your letter and I'm so glad you reached out. It's true that numbers don't lie - but what's been so obvious to so many of us (especially over the past couple of years) is that it is so easy to use numbers in order TO lie! This happens in the news and in advertising ALL the time. That's why it's so so important to learn and understand math at any level, so that you can empower yourself with enough confidence to say "hey - those numbers don't look right!"

I'm really excited that you're interested in physics and have no doubt that you can gain an understanding of the concepts that intrigue you most. You're right, there are certain math concepts that make understanding physics a whole lot easier. If you're looking for a place to start, I would go with basic algebra - algebra is the alphabet of physics and will give you the foundation you need to start understanding the physics concepts that interest you! Physics also uses a lot of concepts found in high school calculus, so once you're feeling good about algebra, you could head that way next! Above all, follow your curiosity. If you're interested in physics, I honestly think you could pick up an intro to physics book NOW and start learning some pretty cool stuff. Doing that, in conjunction with beginning to learn basic algebra, will set you off on a pretty solid journey. You've got this - can't wait to see what you discover!

Got math anxiety? Think you're a hopeless case? VaneSSa to the rescue! The Lady Gaga of mathematics will put her Master's in Mathematics Education to good use by delivering the ultimate personalized pep-talk!

vanessa@themathguru.ca

Prison Mathematics Project,
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INSIDER INSIGHTS

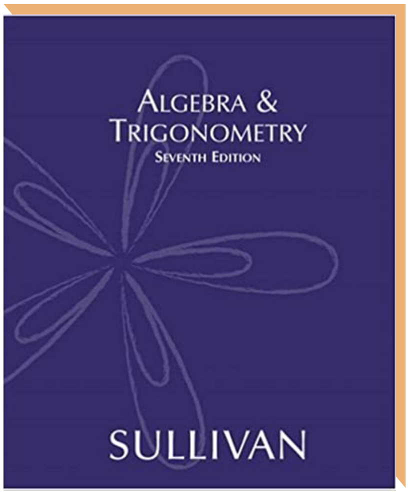
In this column, we publish feedback from PMP participants, mentors and volunteers. Do you have something to say that will educate, encourage, entice or entertain your fellow math students? Drop us a line at the address on the back page, or if you have access to email, send a message to PMP@pmathp.org. We welcome your input.

Here are some pointers from Participant Billy Perez

"In your last newsletter, an inmate asked your advice columnist how best to pursue algebra self-instruction while incarcerated. I have some concrete suggestions.

First, have someone send you a decent used algebra-trig textbook from Amazon Used, thriftbooks.com, textbookrush.com, etc. I recommend Sullivan's *Algebra & Trigonometry* (older editions are best--we have no use for any pointers to web content, programmable calculator exercises, etc.).

Second, do not start by reading. Start by doing. Go straight to the exercises and try to solve the first one. If you can't make heads or tails of it, read JUST ENOUGH of the chapter to allow you to tackle the first exercise. Immediately check your answer. If you get it wrong, go back to the text and see what you're missing, and try again. If you get it right, do the next problem (which probably won't have the answer in back since it's even-numbered--you can usually check it yourself by plugging it back into the problem, which is also good practice at manipulating expressions). ALWAYS check the answer to each problem IMMEDIATELY after finishing the problem. Immediate feedback is essential. Don't do several exercises and then check them all. Solve & confirm one by one. If you write your work down, each step separately, you can more easily spot where you went wrong.



Third, keep solving exercises until you reach one you don't understand. At that point, read the chapter text where you left off, reading only as far as is necessary to solve the current exercise, and continue.

Fourth, when you're done with the chapter's exercises, go back and read any portions you didn't finish. Then begin the next chapter the same way--with its exercises.

And, of course, if you get irretrievably stuck, reach out to PMP."

Billy Perez

Rory Andes

PARTICIPANT SPOTLIGHT

by Claire Finlayson

Well, someone has to keep these math geeks in line and jump on every little spelling mistake... I'm a writer from BC, Canada, so don't try to bust me for using British spelling, like "cheque" and "colour," okay? I am a huge fan of the PMP in spite of my rudimentary math skills, so how could I help? Well, I'm a storyteller at heart and nonfiction is my jam. It's the unique true-life stories of people that interest me. So I write profiles of participants and volunteers and others involved in this program.

Claire Finlayson

If there's someone you'd like me to spotlight, here's how you can contact me:

www.clairefinlayson.com

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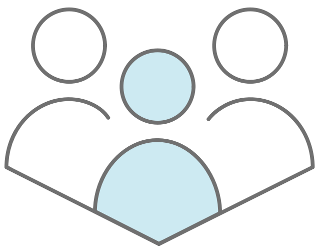
I am interviewing Rory Andes, an inmate at MCC/TRU in Washington State serving an 11-year sentence and due for release in October 2023. Rory is a 12-year veteran of the US Army including two tours of duty in Iraq. I have heard it said that prison can be either a tomb or a womb, and in Rory's case, it has obviously been the latter. He continues to do extensive work to rectify past mistakes and deal with his complex PTSD. He is recognized as a public speaker and executive leader with Toastmasters. He's a peer counsellor, a prolific writer on www.humanme.org and has collaborated with the University of Washington on prison reform initiatives. Oh, and he makes quilts for Seattle's homeless.

CF: Rory! Hello! You have told me that after your incarceration you finally became "free." That's an intriguing concept. Can you say more about that?

RA: My life prior to incarceration was absolute chaos. While it looked relatively balanced to most, I was a combat vet struggling with unseen traumas. I had come home from Iraq to a pile of trouble at home, including a failed marriage. The struggle to find a meaningful place in life was overwhelming. Like a lot of people that end up in prison, I was hanging by a thread, and I crossed a lot of my own moral boundaries to find anything that looked like acceptance. I guess at that point in my life I was like a man trying to brush his teeth without arms -

it can be done, but I certainly didn't know how, and even the simplest actions often seemed impossible. But when I came to prison I realized how to manage those traumas, and finally got the right support for the things that had ground me down. I learned that asking for help was a good thing and found community in people just like me who had experienced adverse life events and had risen above them. There's freedom in being a contributing member of society (yes, including prison society) with trusted friends and colleagues. I began to see the world through a newly polished lens. Prior to prison, I was shackled by my own mind. I only broke those chains by being here.



CF: Wow. I'm going to have to rethink my stereotypical ideas of prison as an exclusively bad place that creates "better criminals," because it clearly hasn't been that for you. Speaking of being "free," I've heard that you were under an extended covid lockdown at MCC until recently. Is it true you were all confined to your rooms 23 hours a day for 50 days? How did you cope with that?

RA: Yes, our prison was the subject of a covid outbreak that created some pretty brutal conditions—both for those who became infected and those who didn't. In an effort to slow transmission of the virus, all inmates were separated into cohorts and locked for 23 hours a day in our 8x12 cells. It wasn't intended to create more hardship, but it did. We were only allowed out with our cohorts (15 people in mine) for one hour per day—our only chance to shower, clean our cells, or even phone home. There's something to be said about the psychological stress resulting from that kind of confinement, so it's with good reason that prisons across the nation are trying to rethink isolation and segregation. Just as the average citizen feels overwhelmed by being quarantined in their homes, prisoners feel that just as much and sometimes worse. The inactivity, the isolation and the lethargy wear deeply on a person's connection to life. It's really depressing. However, we all pulled through.

CF: Ugh, I can't imagine how hard that must have been.
Listen, I know you have a short time left on your sentence. You have often spoken about being released from prison as being something like returning from war. That's a little daunting! Can you elaborate on that?

RA: Releasing *is* like returning from war, and that fact drives my future plans. The way I see it is that in combat, you are put in some awful situations and you learn to rely on others to see tomorrow. A prison community is in some ways like that. You collectively carry the nuances, the lexicon and the weight of it, just like guys in combat together learn to do.

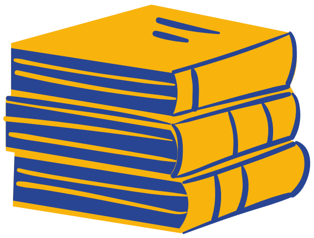
Soldiers can experience situations and traumas that their families could never understand. But then one day, it's time to go home. What then? The most successful ones have a supportive structure in society filled with people who "get it." For prisoners the same thing is important after release. There's a lot that gets missed in American culture about the similarities of the traumas both soldiers and prisoners endure, but it doesn't change the reality. Having been both myself, I think I can help those releasing blend back into free society with clear focus and understanding of mental health issues and community-based resources. When I came home from war, I just wanted someone to accept the broken parts in me.

CF: I think you've drawn a very apt comparison. Do you have any plans in place for after your release?

RA: My goal is to work in a Human Services field helping to guide inmates out of prison and back to society again. I want to be part of a non-profit, social justice or college reintegration organization and be that guiding light to show them acceptance and a path to living well, while rising above their adverse histories.

CF: That is such a practical way to atone, to give back. Who better than a combat veteran who has "done time" to work in this field? Talk about someone who "gets it"!

RA: Several years ago, I was part of a select group who collaborated with the University of Washington. We hosted a summer class for free-world UW students to provide a first-hand account of inmate issues. One result of this collaboration was the need for a peer-based re-entry program developed by us, for us. So we did that, and as we grew, we took on community stakeholders, and now we counsel releasing inmates, providing a seamless transition plan for pre- and post-release. We educate inmates on available resources in prison and our stakeholders ensure the availability of those resources out there. The real genius of the PMP is that it absolutely fits in with this concept. I was even able to coauthor a curriculum designed to instruct on certain fundamental life skills like resume writing and budgeting. To say it's extremely fulfilling would be an understatement. It's become my life's work.

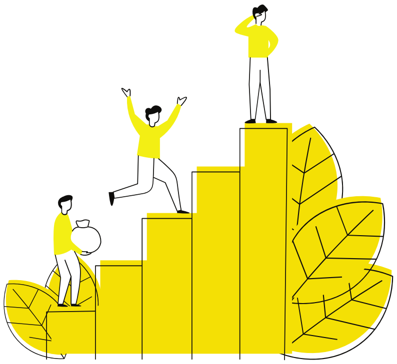


CF: How I love hearing that! You're not waiting to get out for your life to begin – merely to continue on a path you have already begun to walk while in prison.

That's certainly my impression from reading your work on the blog humanme.org. You are all about positive change and rehabilitation – in fact, I've been chastised by both you and Christopher for using the word "con" in fun. Apparently it's not funny, as none of you incarcerated people in the math community buy into the "convict code." If I didn't know this, there are lots of other folks who don't. Can you elaborate?

RA: Yes, the word "con" is a moniker for those who've surrendered to the prison culture. That's not me. I'm absolutely against the "convict code." That code allows the exploitation of one's prison community. Plenty end up in prison because they've exploited their community, then some choose to continue doing the same in here. The code says there's nobility in bullying or beating another person for a perceived weakness. The code provides a stratum for who has a "better crime." The code insists that collaboration between prison administrators and prisoners can't exist. The convict code is flat wrong. To hold the title "con" means someone has failed at their own humanity, because none of these things are true. No thanks! And let's face it, how many "cons" are invested in academia? Certainly none that I know.

I'm a citizen awaiting return to the society that made a decision about my previous actions, and when I'm done, I intend on being a good neighbor, a civic-minded community member, and a success.



CF: Thank you. That's clear to me now. So is the fact that you have threatened to turn me in for jaywalking if you ever get the chance... You've scared me straight, Rory!

Your support system seems to consist mostly of your math friends in prison. I happen to know that two of your closest pals are out now. How will you sustain those relationships going forward? And do you have any other support in place outside of prison besides those who became your friends inside?

RA: I have been blessed to include people in the “in house” math community as some of my best friends. People like Christopher Havens, Marshall Byers and Ruth Utnage have become like family to me. I became charged by our synergy and common desire to be better people in here, and you can't ignore what that does to a guy's soul. I firmly believe that you are the company you keep and I'm in some amazing company. As my friends leave prison, I rely on them to be the reality check for me as to how my own release planning is going. They are my community support. I stay in touch through writing and phone calls to the free world. Because of covid, making connections within the prison has had to be a bit crafty, but we make it work. When it comes to collaboration, I'm reminded of a saying I heard years ago that *"If necessity is the mother of invention, prison is surely its bastard father."* The connections inside can be a challenge, but the power in my friendships with Christopher and other math minds in here are far beyond the math itself and worth the efforts.

CF: Now, ahem, since this is a math newsletter, we should talk more about the Prison Mathematics Project. Can you please tell me how you got involved, and what this program has meant to you personally? I mean, did you always have an interest in mathematics? Or is this something new that you’ve picked up by osmosis being around Christopher?

RA: Math has always been a bit of a monster for me. (The Ask The Math Guru section of this publication by VaneSSa Vakharia is absolute perfection in that regard!) My area of interest is college-level algebra for my degree path.

I've been friends with Christopher for a long time, and I have watched him take on the fostering of other mathematical minds over the years. He has invited me to help in math events he has hosted at our prison, and as I stated previously, it's the synergy of these like minds that becomes a compelling force for good. Math can be a vehicle for reshaping human values. I may not be a big math guy, but we share the value of progress and productivity.

When Christopher told me about the PMP, I jumped at the chance to get involved because I have an interest in higher education. I was paired with a medical student from Canada named Cole. He’s an amazing person to collaborate with, and a key to my own sense of humanity. The conversations we have about medicine and the human condition are far greater than just those about the elements of quadratic equations.

And that's the power of the PMP—the connection to a world so much bigger than prison. Things like that don’t happen without the PMP volunteers. I think any prisoner who has lost it all—with the exception of their intelligence—as a result of their incarceration, can be absolutely revitalized by a PMP volunteer. Yes, the PMP means a great deal to me personally on so many different levels. It’s all so beyond amazing!

CF: Yes, it is. I appreciate your taking the time, Rory, especially since you’re finally out of lockdown and up to your eyebrows in work and school. You truly are an inspiration. We will look forward to following your further adventures after your release.

THE PRISONER'S DILEMMA

Glen Whitney, AKA The Problem Warden

Starting with this newsletter, the problem column is growing, and we need your help. This time, we have two new dilemmas for you to crack, and we're still looking for solutions to Dilemma 2 from last time. So send in your responses, complete or partial, and all will be credited in the next iteration of The Prisoner's Dilemma. We're looking for proposals for new dilemmas to appear here, too. If you've created a problem that's really gotten your mind racing, pass it along. We'll also connect with the math community around the world through the on-line version of this problem collection at prisonersdilemma.org. Full submission guidelines are at the end of the column.

D3: Zero Sum Game

Contributed by Ian Stewart, University of Warwick

Is it possible to number the edges of a cube using each of the numbers -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, and 6 once, so that for every vertex, the sum of the numbers on the edges that meet there is zero? (See figure 1 for a diagram of the cube showing where to fill in the numbers.) What about if instead you use the 12 consecutive integers from -5 to 6, inclusive?

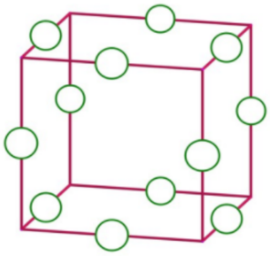


Figure 1: Number all edges to make the sum of those meeting at each corner zero.

D4: Four Operations Revisited

One of the great things about mathematics is that it's always possible to look again at familiar things and discover something new. Sometimes this can happen when you combine well-known things in a new way. For example, you're probably familiar with the four basic operations of addition, subtraction, multiplication, and division. And you may have heard about various number sequences, such as the powers of two: 1, 2, 4, 8, 16, and so on. In that sequence, you obtain each term by adding the previous term to itself, or in symbols $a_n = a_{n-1} + a_{n-1}$. Another similar sequence is the sum recurrence, also known as the Virahanka or Fibonacci sequence, in which you obtain each term by adding the previous term to the one before it, or $b_n = b_{n-1} + b_{n-2}$. The sum recurrence begins with 1, 1, 2, 3, 5, 8, and so on.

This dilemma challenges you to describe the long-term behavior of four different sequences. Unlike sequences *a* and *b* which just add one of the last two terms unchanged to the previous one, each of these sequences uses one of the four basic operations on the last two terms to determine what to add to the previous term. But first, what is meant by "long-term behavior?" The gold standard is a closed-form formula for the sequence – an expression for the *n*th term that just involves *n*, not previous terms. For example, the closed form for the powers of two is $a_n = 2^n$. If there doesn't seem to be a closed form, maybe you can find a formula that works when *n* is bigger than some number, but might be off for the first few values. Or sometimes you can find a formula that is never exactly right, but gets closer and closer as n gets large. An example of this possibility is the formula $b_n \approx \phi^{n+1}/\sqrt{5}$ for the sum recurrence, where $\phi = (1+\sqrt{5})/2$ is the golden ratio. All else failing, it could be one formula that's always less than the sequence and another that is always greater – and hopefully you can get those two “bounding” formulas as close to each other as possible.

So all that said, can you describe the long term-behavior of each of the four following sequences? Each sequence starts with 1, then 2 (like *a* and *b* do), and thereafter adds the described quantity to the previous term to obtain the next term. If the described quantity is ever not a whole number, it should be rounded up to the next whole number.

c) One less than the sum of the previous two terms. So to obtain the third term of this sequence, you take 1+2 - 1 = 2, and add that to 2, to get 4.
d) One more than the difference of the previous two terms. This time, the amount you add to get the third term is 2-1 + 1 = 2, so the third term is again 4. (But don't worry, this sequence will veer off from sequence (c) soon!)
e) Half the product of the previous two terms. So in this case our next increment is $2 \cdot 1/2 = 1$, and the third term will be 3.
f) Twice the quotient of the previous two terms. Now the next increment is $(2/1) \cdot 2 = 4$, and the third term will be 6.

Which of these four sequences ultimately grows the slowest? You will see that by using each operation as part of an infinite cycle, we can get new, different behaviors. Happy sequence hunting!

D1: Continuing Irrationality

(*First appeared Spring 2021*)

One interesting thing about continued fractions is that every infinite continued fraction represents an irrational number. But in our first dilemma, we asked what happens when you have an infinite sequence of finite continued fractions with more and more terms. Recall that the binomial coefficient $\binom{a}{b}$ has the formula $\frac{a!}{b!(a-b)!}$. Now for any integer $m \geq 0$, define the m+1st term in our sequence by

$$C_{m+1} = \binom{m}{0} + \cfrac{1}{\binom{n-1}{1} + \cfrac{1}{\binom{n-2}{2} + \cdots + \cfrac{1}{\binom{n-k}{k}}}}, \quad \text{where } k = \left\lfloor \frac{m}{2} \right\rfloor.$$

(The notation $\lfloor x \rfloor$ means “the greatest integer less than or equal to x,” so $k=m/2$ when m is even and $k=(m-1)/2$ when m is odd.)

For example, we have

$$C_6 = \binom{5}{0} + \cfrac{1}{\binom{4}{1} + \cfrac{1}{\binom{3}{2}}}.$$

SECOND HEARING

This is the portion of the column in which we revisit prior dilemmas, to share solutions that readers have come up with, or report on their status. The problems are reprinted, sometimes in abbreviated form, so that you can easily follow along.

Although most of our problems are new, published here in The Prisoner’s Dilemma for the first time as testament to the fact that math is always unfolding and presenting new challenges, some questions are so intriguing that they deserve another look. These are the “classics” of the problem world, and our second dilemma was among them.

As n becomes infinitely large, that is, as $n \to \infty$, the continued fraction C_n has more and more terms.

Does that mean that C_n approaches an irrational number as $n \to \infty$? What is $\lim_{n \to \infty} C_n$, if it exists?

D2: Find Yourselves

(*First appeared Fall 2021*)

In this classic dilemma, the director of a prison offers 100 prisoners, who have been assigned numbers from 1 to 100, a last chance at freedom. A room contains 100 boxes. The director randomly puts one prisoner's number in each closed box. The prisoners enter the room, one after another. Each prisoner may open and look into 50 boxes in any order. The boxes are closed again afterwards. If, during this search, every prisoner finds his number in one of the boxes, all prisoners are pardoned and given $5,000 each. If just one prisoner does not find his number, all prisoners are admonished and never get another chance with this dilemma.

Before the first prisoner enters the room, the prisoners may discuss strategy — but may not communicate once the first prisoner enters to look in the boxes. There's no way for them to always win, but what is their best strategy?

The Prisoner's Dilemma has yet to receive any submissions attempting Dilemma 2, so we're holding it open until next time. Remember, even if you aren't sure you've found the best possible solution, we want to see the progress you've made, so send in the best answer you've come up with.

From the Problem Warden

I am delighted to become part of the Prison Mathematics Project through The Prisoner's Dilemma, as its (honorary) "Problem Warden." My love of mathematics – and especially sharing the joy it can bring – have been a part of almost everything I've done as a professor, financial analyst, parent, founder of the National Museum of Mathematics, author of the Studio Infinity blog, and the most recent ex-editor of The Playground (the problem column of Math Horizons, a magazine chronicling the world of math for the Mathematical Association of America). I look forward to seeing all of the new methods and creative questions that you come up with as we face the many dilemmas to come, together. And don't worry, as a one-time specialist in mathematical logic, this Warden will keep a sharp eye out for any infractions in your reasoning! **– Glen Whitney**

Solution to D1:

We received a submission for this dilemma from PMP member William Burns, who correctly identified that the limit of this sequence is **one**, *not an irrational number. So although a single infinite continued fraction must represent an irrational number, a sequence of ever-longer finite continued fractions can tend to a rational number. It appears that there may have been other submissions for this problem, but I must apologize for not being able to track them down. So if you sent something in, please let us know and we'll credit your contribution next time.*

Here's why $\lim_{n\to\infty} C_n = 1$.

Notice that the leading term of C_{m+1} is $\binom{m}{0}$, which is always equal to one. So for $m>1$, C_{m+1} is one plus something, and that something is always positive. And the second term of C_{m+1} is $\binom{m-1}{1}$, or just $m-1$, so the quantity that we are adding to that initial one is a fraction with numerator 1 and denominator $m-1$, possibly plus something positive. In any case, the quantity added is less than $1/(m-1)$. In other words, we have shown that for $n>2$,

$$1 < C_n < 1 + \frac{1}{n-2}$$

(we have $n-2$ in the denominator here as n and m differ by one because of the definition of the sequence C.) But as $n\to\infty$, $1+1/(n-2)\to 1$. So by the Squeeze Theorem for sequences, we must also have that $C_n \to 1$. ■

Submission Guidelines

Solutions to problems published in *The Prisoner's Dilemma*, and proposals for new dilemmas, are welcome. For solutions, please clearly indicated the dilemma number being solved. If a problem has multiple parts, you may submit solutions to any individual part or parts. Solutions to the dilemmas in this newsletter must be received by the deadline of 2022 May 27. Dilemma proposals will be considered on an ongoing basis. All submissions should be addressed to Glen Whitney/Prisoner's Dilemma either by email at dilemma@pmathp.org (in which case PDF format is preferred, if possible, although any reasonable format will be accepted), or by mail at:

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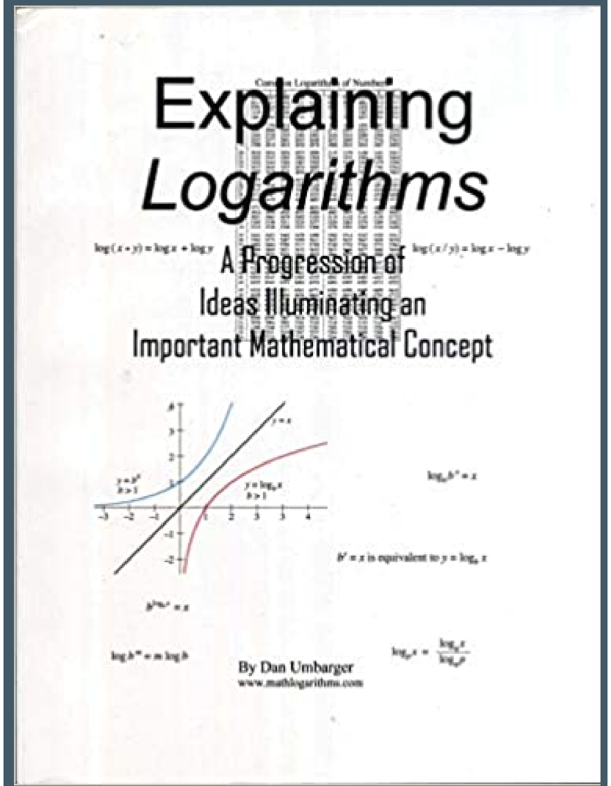
BOOK REVIEW REVIEW

Our featured book is: ***Explaining Logarithms – A Progression of Ideas Illuminating an Important Mathematical Concept***, by high school mathematics teacher Dan Umbarger



No, it's not a typo! We actually review your book reviews!

Here's how it works: You read the featured book and submit a review (details below). If your submission is deemed most inspiring to potential readers by our panel of judges, it will be printed in a future edition of the newsletter, and on the PMP website.

Explaining Logarithms explains not only logarithms but their relationships to other core mathematical ideas.

For over 350 years, most of the mathematics done by scientists, engineers and astronomers was assisted by logarithms. The logarithmic technique was developed to aid in the drudgery of simplifying long, difficult and tedious arithmetic expressions to a lesser level of difficulty. Scientific calculators have made much of the pre-1970 precalculus curriculum obsolete. But the use of calculators has also come with a price: the instruction of logarithms today is much more condensed and abstract than it used to be. As a result, many of today's students do not achieve the same level of "internalization" of logarithmic concepts and ideas. Many do not understand the "mathemagical" formulas they are taught and asked to manipulate.

As you proceed in your study of science, engineering, economics, etc., you will encounter many situations where it is necessary to solve for an unknown variable exponent. Traditional math books will present intimidating math identities and show you how they can be used to solve complicated equations where the exponent is unknown. However, using these mysterious log identities without knowing where they come from, without knowing what justifies them, sets you up for eventual confusion. The book *Explaining Logarithms* will teach you how to solve for unknown variable exponents and where the mysterious log identities come from.

Want to participate in this contest? Here's how:

Reviews must be no longer than 1,000 words and all submissions for the review of *Explaining Logarithms* should be submitted by June 15, 2022.

If you can't obtain the featured book on your own, we'll be happy to send you a copy while supplies last. Simply write and let us know about your circumstances. The trade-off is that we'll be expecting a review from you.

To submit electronically, send your review to:
PMP@pmathp.org

Mail paper submissions to:
Prison Mathematics Project
Attn: Book Review Submission
10810 N. Tatum Blvd, Suite 102-998
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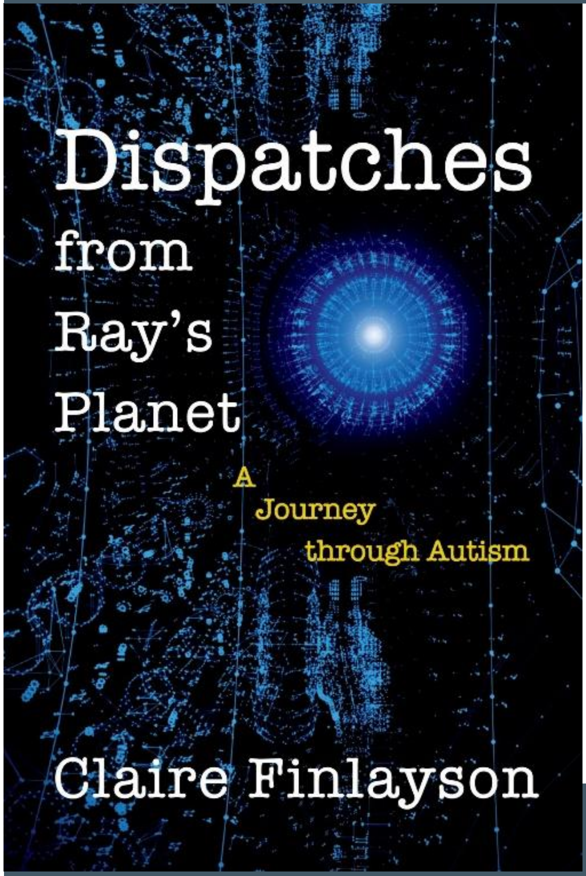
BOOK REVIEW REVIEW WINNER



Our Fall 2021 featured title was *Dispatches from Ray's Planet: A Journey through Autism* by our very own newsletter editor, Claire Finlayson, who, she assures you, is not biased in the least when it comes to rating reviews of her book. ☺ (And neither is Ray, who is also a judge.)

Although not a math text, *Dispatches* showcases Ray's patience and dedication as a math tutor and his profound reverence for mathematics, "the language in which God wrote the universe." It also shows the value of a mentor/mentee relationship and what a little confidence can do for a struggling student—all this wrapped up in a very human story told with empathy and humour. This book was the catalyst that led to Ray's involvement with the PMP (and Claire's, too).

We would like to thank all who wrote in with reviews of *Dispatches*. Our judges had a hard time choosing the winner, as each review reflected the style and personality of the writer and their individual take-aways from the book.

> **I can't possibly choose. These reviews are all deeply personal gifts and I will not rank them. That's rude on both our planets. Or should be. – Ray**

Ray's objections notwithstanding, the judges have made their decision.

First runner-up:

DANIEL JOSIE, Greencastle, IN

Daniel's review had a great opening paragraph and the use of quotes from the book were highly effective in illustrating Ray and his social challenges. This review will be posted on the PMP website.

And the winner of the review contest for Dispatches from Ray's Planet is:

JON HARRIS, Monroe, WA

We are pleased to publish his review here.

Goongbalong: The Game of Social Minutiae
By Jon Harris

In *Dispatches From Ray's Planet*, Claire Finlayson gives a powerful and in-depth look at her struggle to understand Ray, her autistic brother. Far from the sterile and unfeeling academic publications written on Autism Spectrum Disorder, *Dispatches* is a warm, human, and realistic journey through the lives of these siblings as told by their regular correspondence.

Ray's family always knew that he was different, and these differences made their interactions with him confusing, complicated and sometimes painful. Though obviously of great intelligence, Ray struggles to understand the most basic of social concepts. These notions of common courtesies and etiquette, part of a never-ending game that Ray calls Goongbalong, seem impossibly complex and arbitrary to him. He is constantly making simple mistakes, or Goongbalong fouls, that alienate, infuriate, and bewilder those around him. It isn't until later in Ray's life that his sister picks up a book on Asperger's syndrome that things start to make sense.

As Ray has always had an easier time writing down his thoughts than he does speaking them, he and his sister begin exploring this new avenue of understanding through the writing of letters.

Ray's difficulties with those things that come naturally to others are tragic, but his triumphs are profound. From his love of math and teaching to his passion for free-diving and astronomy, Ray's life comes out in stunning detail as told by his sister and their letters.

This first book of Finlayson's is an absolute triumph. Aside from being unusually entertaining, it is heartwarming and educational. Without a doubt, Dispatches is a must read for anybody striving to better understand the human condition and those who fall outside of its baseline.

Mail paper submissions for all book reviews to:

Prison Mathematics Project
Attn: Book Review Submission
10810 N. Tatum Blvd, Suite 102-998
Phoenix, AZ 85028

Judges' comments:

Jon Harris wrote what would have been the perfect back jacket copy for my book. It is articulate and insightful. He relates to Ray as only a fellow spectrum-dweller can. Two thumbs up. **– CF**

***

Readers seem to respond viscerally and positively to *Dispatches from Ray's Planet*, universally recognizing something in themselves, or in the world around them, from the book.

Daniel Josey identifies the book as a distillation of some of the research pertaining to Autism Spectrum Disorder, but he also endorses it simply as "a good story about family and overcoming obstacles." In his review Josey homes in on the challenge of switching perspective to an outsider's view. Josey selects pertinent passages from *Dispatches* that draw the reader in. He highlights examples from the book that would intrigue the prospective reader. Josey's review is a thoughtful and reasoned endorsement of the book.

Jon Harris provides lavish praise for *Dispatches from Ray's Planet*. He summarizes the challenge Finlayson took on: to try to understand her brother through the lens of Autism Spectrum Disorder. Harris's review makes us identify with Ray, the subject and protagonist. He makes us want to cheer him on in his struggles. Harris sticks his neck out with the emotional positivity of his review.

Of these two fine reviews Jon Harris's would most tempt me to read *Dispatches from Ray's Planet*. **– DG**

***

Each of the reviewers succeeded in making the case that *Dispatches* would reward the reader with a story at once both human and educational. I will certainly read this book now!

Of all the reviews, the one by Jon Harris had the greatest impact on me. He caught my interest early by clearly stressing that this book was definitely not an "unfeeling academic publication" on ASD or Asperger's syndrome. That interest was held when he covered some of the ways in which the book reveals how Ray's world differs from ours. Harris's review inspires the reader to imagine for a moment what navigating a Goongbalong world would be like. This reviewer succeeds in introducing us to the very human story of the journey now being taken by the siblings, Claire and Ray, and how their letters reveal the richness, the challenges and the complexities of Ray's life. **– MB**

***

It is clear from all of the reviews I read that Claire's writing, Ray and his story, and the evolution of their relationship touched the reviewers deeply. It's very hard to pick my favourite. I truly valued each of them. In terms of a straight book review, I think the one by Jon Harris is the most well-written, ticking almost all of the boxes I would be looking for in a review. **– JL**

Dr. Amy Shell-Gellasch

MATHEMATICS FROM ANOTHER TIME AND PLACE

Dr. Amy Shell-Gellasch is a full time lecturer at Eastern Michigan University. She earned her DA in mathematics from the University of Illinois at Chicago in 2000 and followed that with a post doctorate position at the United States Military Academy at West Point, NY. Her area of research is the History of Mathematics and its uses in teaching. She co-founded and currently chairs the History of Mathematics Special Interest Group of the MAA and is an associate editor of Convergence online journal. Currently she is the Chair of the Michigan Section of the MAA. She conducted research on mathematical devices at the Smithsonian National Museum of American History from 2012-2017.

Multiplication in Ancient Egypt

Hello, I am an historian of mathematics. Yes, that is a thing! Mathematics is as old as time, almost. Did you know that the earliest known "writings" in the sense of deliberate markings made by humans, are mathematical in nature? They were, what we would call tallies. We don't know what the sets of marks or tallies represent, perhaps a shepherd's herds, a simple calendar, or some other count, but they are clearly a record of amounts of something and date back at least 30,000 years ago in Africa.

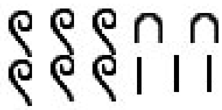
How do you become a math historian? You read, A LOT! You need to understand both mathematics and history. You can't understand how mathematics was done in the past if you only try to compare it to what we do today. You have to understand the people and culture to be able to understand how the mathematics was conceived and performed. I have a doctorate in mathematics, and a strong background in history and historiography (and science). My career has been spent mostly in researching mathematical devices (I was even a researcher at the Smithsonian for several years) and on incorporating history into the mathematics classroom.

One topic that has intrigued me is the different ways to count and calculate. These vary widely from place to place, and throughout time. Yet there are also strong similarities between these different approaches. For example, many methods for multiplication revolved around doubling. So that is where I will start, the doubling and halving method of ancient Egypt.

Many people intuitively use doubling to multiply. When my son was quite young, he asked me what 5 x 8 was. Like any parent who happens to be a math teacher, I suggested he try to figure it out. He said, "Well, twice 5 is 10, and twice 10 is 20, and twice 20 is 40. Is it 40?" I was over the moon, my son was a genius! He realized that 8 is twice 2 three times, and just transferred that to the 5. This idea was used in many cultures. Egyptian papyrus discoveries, such as the Ahmes papyrus (c. 1550 BCE) which is basically a math textbook for scribes, show multiplication and division problems and well as area and volume calculations involving whole and fractional numbers.

Here is an image of the Ahmes (formerly known as the Rhind) papyrus showing various mathematical problems. The papyrus is currently housed at the British Museum, with a small fragment at the Brooklyn Museum in New York.

I will show you the multiplication and division algorithm for whole numbers, just to give you a taste. Again, it all hinges on the number 2. I will be using our Indo-Arabic numerals as opposed to Egyptian numerals for simplicity's sake. But here is an example just to satisfy your curiosity:

623: 

Let's calculate 49x28. The scribe would make two columns, the left starting with "1" (but they just put a dot) and the right with the larger multiplier. Then they would double each column until they could find numbers on the left that added to the smaller multiplier. Then the sum of the corresponding numbers on the right would be the desired product. Here is what it would look like.

49 x 28

	.	49
	2	98
\	4	196
\	8	392
\	16	784
	28	1372

The red marks on the left denote the doubles of 1 that add to 28. Adding the numbers on the right that correspond to those marks gives us 196+392+784 = 1372, the correct product!

Will is always work?

YES!

Why does this work?

Hint, think computers. We have in essence converted the 28 into its binary representation, and then added up that many copies of 49.

Now let's use this idea in reverse to divide. We will use the same example so you can see the process unwind, but you can try this on your own with any known product you like. Note that if the division is not exact, the process can be continued to get a fractional remainder to as much accuracy as desired. But since the Egyptians did not have fractions as we know them (they only had unit fractions like 1/3 and 1/16), we will not go into that here. It can get a little hairy!

We start the same way, with "1" on the left and the divisor on the right. Double both columns until you find number on the RIGHT that add to the dividend. Again we denote the rows we need with a red mark on the left. Then the sum of the corresponding numbers on the LEFT will be the quotient.

1372÷28

\	.	28
	2	56
	4	112
	8	224
\	16	448
\	32	896
	49	1372

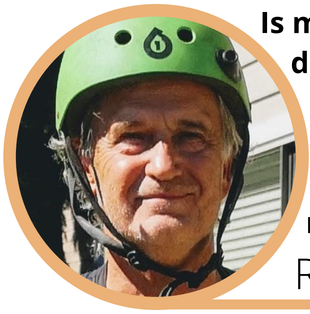
Since 28+448+896 = 1372, then 1+16+32 = 49 gives us our answer! Again it is using the binary representation, but this time of the dividend, which is used to find how many sets of 28 are needed to get to 1372, which is 49.

Voila, now you can calculate like an Egyptian!

If you would like to suggest a topic or submit a piece for That is So Cool! or Math from Another Time and Place, please email me at amy@pmathp.org

PROBLEM CHILD

Today's deeply philosophical "Problem Child" comes from Ray Andrews in British Columbia, Canada. He asks:



Is math really a disembodied bunch of tautologies floating in its own domain, or is it a hard part of reality?

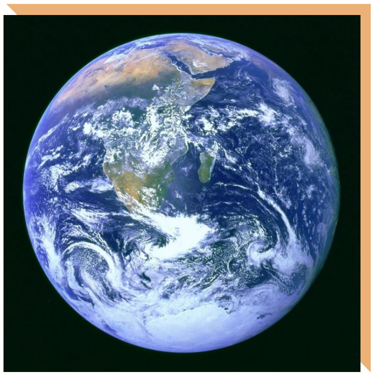
Ray Andrews

We've all been faced with the occasional problem that breaks through our mathematical defenses. In this column, we take your "problem children" and deconstruct them in a way that we hope will aid you in your journey through mathematics.

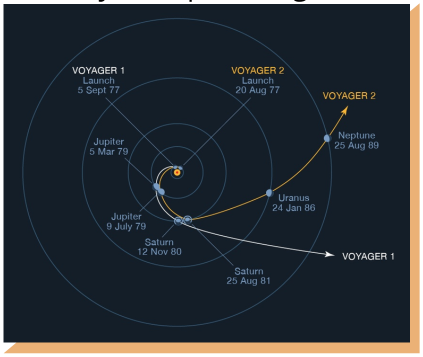
Here's Amit Sahai, professor of computer science at UCLA to answer Ray's question.

Amit Sahai

This is a wonderful and deep question, Ray. The first thing we need to do to tackle it is to define terms. We definitely know that mathematics is at least a bunch of tautologies floating off by itself. But is it more? In particular, is it connected to this thing we call "reality"? And what is reality? This is one of the deepest questions in philosophy, and I won't pretend to try to address it here. But for the moment, let's agree that the external physical world exists and is at least a part of "reality," and let's focus on that part.

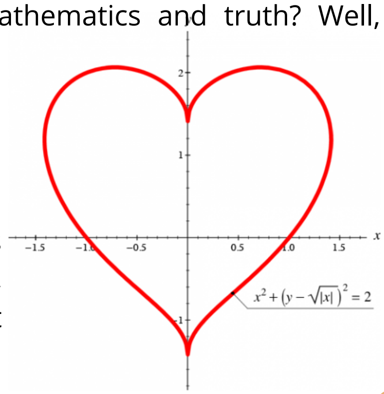


One of the greatest mysteries of this universe is that every single physical law that we have observed regarding matter and energy—every single one—can be described using mathematics. And not only that, but these mathematical descriptions are so remarkably accurate that we can use "pure mathematics" like the mathematical laws of calculus to predict with unbelievable accuracy how the physical universe will behave in the future. It is only because of this awesome predictive power that we could launch Voyager 1 and Voyager 2 and trace out their orbital acrobatics years in advance, and have them meet up with multiple planets using only a minuscule amount of fuel. The intricacies of microchips behave according to the mathematical laws of quantum mechanics and electromagnetism. We have confidently built layer upon layer of mathematics to predict what extremely complex engineered systems will do, and every time—every time!!—the system dutifully follows all the laws of mathematics. I am in awe of this each time I think about it.



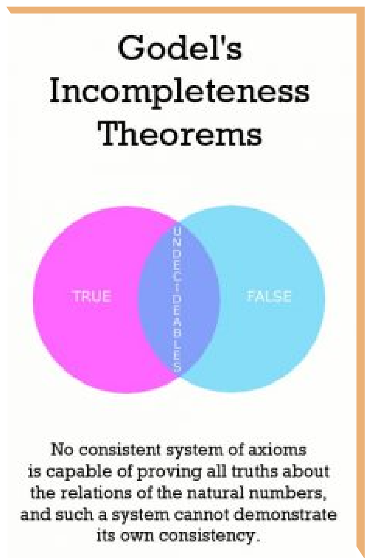
So how do we gain confidence about reality? We run experiments and see if our predictions work. That's why we believe in the law of gravitation, and so many other laws. We aren't 100.0000000000000% confident, but we are 99.99999999% confident (approximately). And I'm sure that when you drop a pencil, you don't wonder at all whether it will fall this time. You never think, "maybe this time it will just float in the air." That's how confident you are in gravitation. Because you have a ton of evidence.
But the evidence that we have for mathematics representing reality is thousands or maybe millions of times more complete than the evidence we have for gravity. That's how reliable mathematics has been for us in understanding physical reality. So if someone asks you, “How confident are you that math has anything to do with reality?”—you have to say, “extremely confident.” Because we literally have more evidence for the incredibly tight relationship between mathematics and physical reality, than we have for the assertion: “The next time I drop my pencil, it will fall down towards the earth.”

I think for a lot of us mathematicians, we believe in this assertion: mathematics is an "undying citadel of truth." We believe that mathematical truth represents eternal truth, completely independent of reality, or even whether reality as we understand it actually exists. I believe that. Why? Honestly mainly because I love mathematics and I feel this to be true in my heart. But what is all this talk of feelings and hearts doing in a discussion about mathematics and truth? Well, unfortunately, I will now argue that such discussions are necessary. And that, when it comes to the truth of mathematics beyond empiricism, there is an inherent component of faith involved. In order to rationally believe that mathematics represents truth, one must either have faith in mathematics itself, or faith in empiricism (or both). As a personal aside, I will confess that I have faith in both.



So why is this true? Let's put empiricism aside for the moment. Can mathematics be "self-supporting"? Well, a minimal condition for mathematics to represent truth is that mathematics be consistent -- that is, that the rules of mathematics don't lead to contradictions. If math were inconsistent, then we are screwed. By the rules of logic, once we reach even one contradiction, then in fact all mathematical statements are true. 0=1. Basically, everything falls apart.

So how can we be sure that mathematics is consistent? Can we somehow prove that mathematics is consistent? Note that even if we could prove using mathematics that mathematics is consistent, this would still require us to believe that mathematics is consistent, because of course if mathematics were inconsistent, then you could prove that mathematics is consistent, because you can prove that 0=1 and therefore you can prove any statement. But still, it would surely be nice if we could prove that mathematics is consistent. Is it too much to ask that mathematics stands up for itself?



Unfortunately, it turns out, it is too much to ask. In his classic and groundbreaking work, Gödel proved (with an additional idea due to Rosser) that no consistent and sufficiently expressive mathematical theory can prove its its own consistency. In my opinion, the best way to understand this result -- known as Gödel's second incompleteness theorem -- is through theoretical computer science and the work of Turing and others on undecidability, and the beautiful Recursion Theorem due to Kleene (and Rogers). Very roughly speaking, this approach shows that if a mathematical theory (based on sufficiently expressive axioms) can prove its own consistency, then it can also prove a theorem that itself asserts the falsehood of the theorem. This is very heady stuff, beautiful and terrifying in its implications.

But indeed, what this means is that mathematics cannot even be self-supporting. We can have an infinite tower of mathematical theories such that the mathematical theory "one floor below" is proven consistent in the theory just above it, but still no theory is actually able to prove its own consistency.

So, if you believe like I do that mathematics is a beautiful bastion of truth, or indeed Truth with a capital T -- more true than, say, the existence of gravity -- then that's great, but it is indeed a confession of your faith in mathematics. I share this faith.

> DO NOT LOSE YOUR FAITH. A MIGHTY FORTRESS IS OUR MATHEMATICS. MATHEMATICS WILL RISE TO THE CHALLENGE, AS IT ALWAYS HAS.
>
> - STANISLAW ULAM -

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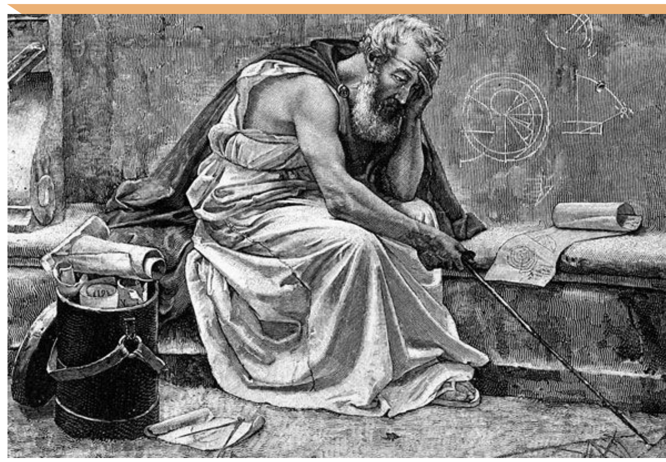
However, if you aren't quite willing to go that far, then it suffices to have faith in empiricism—to have faith that empirical evidence somehow points to truth. If you have faith in empiricism, then you likely believe in gravity, and as we already discussed in Part 1, you should believe that mathematics represents truth in reality, and you should believe that mathematics is consistent -- since we haven't yet encountered any inconsistency despite centuries of mathematical exploration by humanity.



Note, however, that it is always risky to use empiricism to build confidence in a negative statement. We didn't observe evidence of black holes for millennia, but then we did. It could be that there is an inconsistency that is hiding in mathematics, and we just haven't come across it yet. A fascinating possibility -- that I do not believe in -- is that perhaps the mathematics upon which this universe is built indeed has an inconsistency, but this inconsistency cannot be encountered unless one takes at least logical steps starting with the axioms upon which this universe's mathematics is founded.

Then perhaps, one moment far in the future when the universe has allowed logical steps to have been taken, the universe will encounter the inconsistency in mathematics, at which point the universe will spontaneously vanish in a puff of logic. This is fun to think about, but again, I want to be clear that I don't see any reason why this would be. Then again, we can't rule it out either...

Wrestling with a Problem Child? We can help. Our contact info is on the back page of this newsletter.

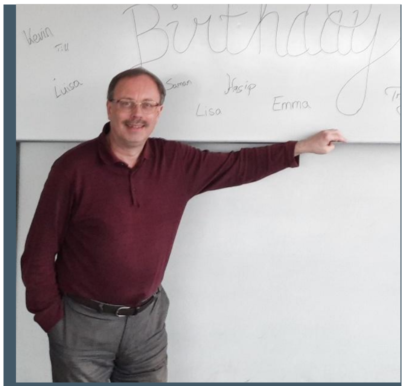
A Miscreant's Miscellany

by Christopher Havens

Christopher Havens

A Serious Talk About Personality Analysis of Numbers

With Carsten Elsner



Carsten Elsner

Numbers are like people: Some are snootier than others and look down upon their brethren. For example, irrational numbers like π feel that they are somehow special because they are not rational numbers; that is, they are not fractions of the shape a/b such as 4658/13947. While fractions can approach irrational numbers at will, some irrational numbers are bitchy and dismissive about their making such an approach, so they expect a great effort from an approaching fraction — a big denominator of the approaching rational number. Other irrational numbers are friendly and inviting; even rational numbers with only a small purse (denominator) are welcome to approach them.

For most people real numbers are all equal to each other, because they have only the real number line in front of their eyes as a line extending from minus infinity to plus infinity and everything looks the same. But Christopher and Carsten know this is not the case, and they use the ultimate tool for the analysis of the personalities of irrational numbers: continued fractions, which reveal how bitchy or inviting irrational numbers can be compared to their simple rational brethren. Here is their chat on the fascinating subject of number personalities.

CH: When you study a specific area long enough, it becomes apparent who the other mathematicians are who share similar interests! Because some researchers within my field of exploration produce work that fits my interests so well, I find myself developing "favorites." Sort of like favorite baseball players. One of these is Carsten Elsner, and he is here with us today. Carsten, why don't you start by telling us a bit about yourself and how you began studying continued fractions?

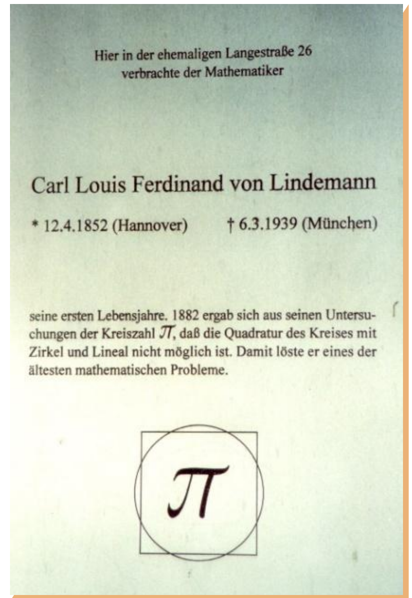
CE: I live in Hannover in Lower Saxony (Germany), where I studied mathematics and physics from 1983 to 1988. In Hannover, the great philosopher and mathematician Gottfried Wilhelm Leibniz spent most of his life in the service of the Guelph princes. We owe him—along with Isaac Newton--the infinitesimal calculus. But here's something only a few people know: Carl Louis Ferdinand von Lindemann, the conqueror of the circular number π, was born in Hanover in 1852. He later proved a property of the number π (the so-called transcendence) in Freiburg in 1882, thus solving the problem originating from antiquity, according to which the squaring of the circle with compass and ruler is not possible according to given construction rules.

Leibniz is remembered everywhere in Hannover (the university bears his name, for example), but Lindemann is unknown to most people, so I started a fundraising campaign for a Lindemann monument in 1997, and I designed it. Unfortunately, Lindemann's birthplace was destroyed in the 2nd World War, but using old city maps in museum archives, I found the original location and was able to inaugurate the monument there in May 1997.

My first "contact" with Lindemann had occurred 12 years earlier, when I studied the transcendence proof for π in an undergraduate seminar. Subsequently, I also learned the methods of proving the irrationality of concretely given numbers α (or of α=π) by suitable approximations of α with rational numbers (i.e., with fractions). In doing so, one is inevitably confronted with continued fractions, a representation of α by a possibly infinite string of fractions, in which each fraction of this chain is in the denominator of the preceding fraction. If one breaks such a continued fraction after a link, one obtains rational numbers that approximate α in a certain sense as best as possible. This theory fascinated me, so I enthusiastically dug into the extensive literature on continued fractions while I was still a student....

Today, however, I work mainly in the so-called transcendence theory (another subfield of number theory, to which Lindemann's transcendence proof for π belongs), where continued fractions do not play a fundamental role. But if I am confronted with problems in the field of continued fractions - as you were - I immediately start to burn for this topic again!

CH: What really got me fired up was number theory. More specifically, it was the wonderful beating hearts of our irrational numbers called continued fractions. And what is a continued fraction? Well, without getting overly technical, a continued fraction is an integer plus a fraction whose denominator is an integer plus a fraction ... and so on. If this nesting terminates, then the continued fraction represents a rational number. But if the nesting continues ad infinitum, then the continued fraction represents an irrational number. A particularly beautiful example can be found in the base of the natural logarithm, e:



$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \ddots}}}}}}}}$$

Continued fractions have lots of wonderful properties hidden from plain sight. In fact, studying these properties is one of my most enjoyable activities. Whenever I spend too much time away from the subject, I find myself yearning to either solve those problems that have been on my list or investigate those peculiar behaviors that always pique my curiosity.

Continued fractions are my playground. I believe everyone has such an area where they can play, whether it be geometry, algebra, or even transcendental number theory. I suppose the reason this aspect of math is so important to me is that when I explore and play in mathematics through my imagination, it teaches me things about myself. In fact this was the catalyst that helped me to search so deeply within myself. Through mathematical exploration I find Truth and Beauty, and from these spring Justice and Love.

Somebody once said that continued fractions are the hearts of irrational numbers. I would add that not only are they beautiful and pure, but they even have a pulse in the sense that their sets of convergents beat to the rhythm of a leaping arithmetic pattern.

Carsten, your work underscores that statement. Unfortunately many of our readers may never have encountered continued fractions, let alone have any understanding of convergents. What do you say we break it down in slow motion? Would you explain for our first timers what convergents are?

CE: Of course. This can best be explained vividly by example.

Consider, for example, the number π. It is an irrational number; that is, it cannot itself be represented as a fraction $\frac{a}{b}$ with positive integers *a* and *b*, but we can approximate it with a degree of accuracy by such fractions. But how can we find such approximating fractions in concrete terms?

One possible procedure is suggested to us by the decimal fraction expansion of π:

$$\begin{aligned} \pi &= 3.141592\ldots \\ \alpha_0 &= 3 \\ \alpha_1 &= 31/10 = 3.1 \\ \alpha_2 &= 314/100 = 157/50 = 3.14 \\ \alpha_3 &= 3141/1000 = 3.141 \\ &\vdots \\ \alpha_6 &= \frac{3141592}{1000000} = \frac{392699}{125000} = 3.141592 \end{aligned}$$

We find an infinite number of approximating fractions which approach the number π better and better. However, is this an effective process? How should we interpret "effectiveness" in this context?

For a quantitative measure of how well a fraction $\frac{p}{q}$ approximates a number like π, in number theory, one must compare the positive distance taken between π and $\frac{p}{q}$ with the size of the denominator *q*. This is where the continued fraction expansion of π comes into play. The beginning looks like this:

$$\pi = [3,7,15,1,292,1,1,\ldots]$$

If we break off this infinite expansion prematurely after the partial denominators 3, or 7, or 15, ... we obtain rational numbers:

$$\begin{aligned} \frac{p_0}{q_0} &= 3 \\ \frac{p_1}{q_1} &= 3 + \frac{1}{7} = \frac{22}{7} \\ \frac{p_2}{q_2} &= 3 + \frac{1}{\left(7 + \frac{1}{15}\right)} = \frac{333}{106} \\ \frac{p_3}{q_3} &= \cdots = \frac{355}{113} \end{aligned}$$

These fractions are the so-called convergents of π and they represent particularly good rational approximations. For the numerators p_n and the denominators q_n of these convergents it even holds that the sequence of distances between $\pi \times q_n$ and p_n converges to zero the closer $\frac{p_n}{q_n}$ lies to π (which means that the index *n* increases). In particular, the distance between π and the convergent $\frac{p_n}{q_n}$ is much smaller than $\frac{1}{q_n}$. The approximating fractions of π previously derived from the decimal fraction expansion do not have this property.

Let's look at a concrete numerical example:

$$0 < p_3 - \pi \times q_3 = 355 - 113 \times \pi < 0.0000302$$
$$\text{and } 0 < \frac{355}{113} - \pi < 0.000000267$$

Any fraction $\frac{p'}{q'}$ that is different from a convergent $\frac{p_n}{q_n}$ (except $\frac{p_0}{q_0}$) and has a denominator *q'* that is less than or equal to *q* results in a greater distance between *π* x *q'* and *p'* than between $\pi \times q_n$ and p_n. Obviously convergents are always in lowest terms, and for each irrational number like π there exist infinitely many convergents. This is only one reason why continued fractions are such wonderful objects in number theory.

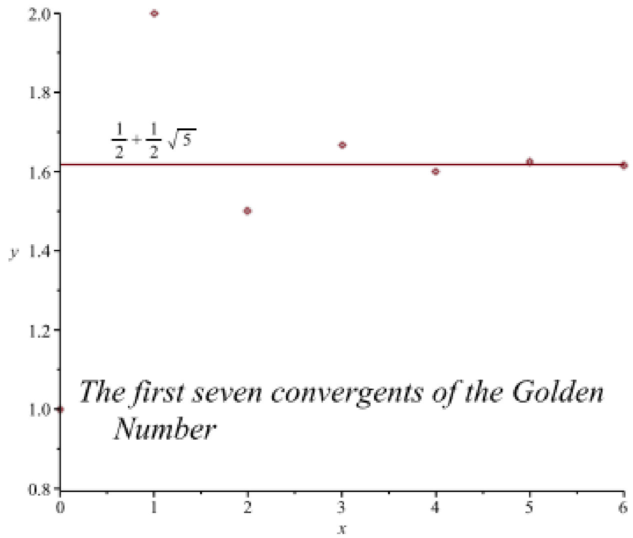
CH: Thanks Carsten! So if I'm hearing you right, the convergents of are basically the "best" approximations to α. That is wonderful!

There is another property which I think is also very beautiful. Specifically for two convergents p_{n-1}/q_{n-1} and p_n/q_n of the continued fraction of a real number α, we have for $n \geq 1$

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}q_n}.$$

This gives us a nice way to visualize the way convergents converge towards the real number α with which they're associated. First, for all odd values of , the difference given above is positive, and similarly, all even values give a negative difference. This means that the convergents $\frac{p_1}{q_1}, \frac{p_3}{q_3}, \frac{p_5}{q_5}, \frac{p_7}{q_7}, \ldots$ lay above α on the *y*-axis and $\frac{p_0}{q_0}, \frac{p_2}{q_2}, \frac{p_4}{q_4}, \frac{p_6}{q_6}, \ldots$ lay below α on the *y*-axis.

Furthermore, since $(q_k)_{k=0}^{\infty}$ is an increasing sequence, those convergents both above and below begin to hug α tighter and tighter! Here's what it looks like graphically:



CE: That's right, Christopher, this is how you find the best approximate fractions above and below the approximated number, depending on your preference, and the index *n* of a convergent, depending on whether it is even or odd, tells you whether the fraction is below or above α.

We can extract another beautiful arithmetic property from the above formula, which relates two successive convergents. Let's multiply this formula by $q_{n-1}q_n$. Then we get

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Because the right side can only take the values -1 and 1, it is never divisible by any prime number. Therefore, the same is true for the left side, and so we read that all fractions p_n/q_n are always in lowest terms. Likewise, it follows from this equation that also the numerators p_{n-1} and p_n as well as the denominators q_{n-1} and q_n of successive convergents p_{n-1}/q_{n-1} and p_n/q_n are always relatively prime.

And the formula for successive convergents you mentioned above holds another exciting possibility. If you sum over this equation by adding up all these equations starting from n=1 separately by left and right sides, you can represent an irrational number α as a rapidly converging alternating series of rational numbers:

$$\alpha = \lfloor \alpha \rfloor + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q_{n-1} q_n}$$

Here $\lfloor \alpha \rfloor$ denotes the largest integer that lies below α. And the numbers q_{n-1} and q_n can be computed recursively: $q_n = a_n q_{n-1} + q_{n-2}$ with $q_0 = 1$ and $q_1 = a_1$. For example,

$$\pi = 3 + \frac{1}{1 \cdot 7} - \frac{1}{7 \cdot 106} + \frac{1}{106 \cdot 113} - \frac{1}{113 \cdot 33102} \pm \cdots$$

CH: There are so many wonderful properties of convergents that I could sit here all day studying this! I hope, though, that any readers who are inspired enough to want to play with these beating hearts of our irrational numbers will reach out to the PMP for some material on the subject. Be forewarned that this area goes fairly deep. It may require a bit of mathematical maturity.

Carsten, thank you so much for sharing your interests around some of these beautiful aspects of continued fractions. Before we go, do you have any advice to give any of our readers who might want to journey further down this road?

CE: Problems in this discipline of mathematics are easy to talk about, but the solutions to these problems are often very difficult. A suitable apparatus must first be built up or deep results from completely different subfields of mathematics must be used as aids. So everyone understands what it means to represent an even number ≥4 as the sum of two primes: 12 is just the sum of the two primes 5 and 7. But in spite of all the sophisticated methods that have been developed in the last 100 years to solve similar additive problems, the question as to whether every even number ≥4 can be represented as a sum of two prime numbers is still unsolved. Today it can only be assumed that it is so.

As you have already mentioned, Christopher, even when dealing with the Diophantine approximation to which the continued fractions contribute, there is this "danger." And indeed: the Fields Medal (the 'Nobel Prize' of mathematics) was awarded to K. F. Roth in 1958 for the proof of a statement about the "bad" approximation of certain algebraic irrational numbers with rational numbers (whatever that means in detail ...). However, Roth did not use continued fractions in his proof.

You ask what my advice is to interested readers who want to learn about continued fractions for the first time. By way of introduction, they could read the relevant chapters in the number theory textbooks, which introduce the basics. No prior knowledge is required other than some matrix calculus—but they should be able to handle formal fraction calculation. Then those who have read up on the basics can opt for a more detailed study of one of the continued fraction specialties. There are, for example, applications for Diophantine equations in analysis, algebra and topology as well as in cryptography. So both you and I found a focus in the Diophantine approximation and have worked a lot on the convergents.

Christopher, I am so pleased that you are interested in my work on continued fractions and contacted me to collaborate. I believe we will have many more interesting and exciting insights into continued fractions. Thank you for this interview.

CLOSING COMMENTS

Christopher Havens

During the pandemic, we here at the PMP have worked tirelessly to reach prisoners across the United States. It brings us great pleasure to inform you that we are now extending our services into Canadian prisons! It is our ultimate goal to reach incarcerated people across the globe so that we can make the world safer through the study of mathematics.

We believe strongly in community, so in an effort to establish deeper connectivity to the math world, the editor of the Prisoner's Dilemma column, Glen Whitney (AKA "The Problem Warden"), has launched the PMP's first problem solvers website: http://www.prisonersdilemma.org

With the support of our volunteer mentors, the PMP has experienced rapid growth, but we cannot continue at this pace without your continued support. Please consider making a contribution using one of the following links:

For one-time or recurring donations: Donate – Prison Math Project
This link allows you to set up easy monthly donations of any denomination:

Donate Now | Prison Math Project (givelively.org)

We welcome partnerships for sharing resources, and we encourage participation from educators who wish to acquire a better understanding of the complexities involved in the teaching and learning of mathematics in highly restrictive environments.

We are looking to expand our team and invite inquiries from those with experience in non-profit organizations. We are also looking for volunteers who can help with the everyday functions of the PMP. Interested parties should send an email to pmp@pmathp.org.

As a show of appreciation to the worldwide mathematical community, we warmly invite you to the annual **PMP Pi Day event** on **March 14, 2022**. This is an all-day virtual event livestreaming at: http://www.prisonmathproject.org.

Thank you for helping us change lives!
Christopher Havens,
Co-founder and CEO, Prison Mathematics Project